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PRIME RINGS WITH PI RINGS OF CONSTANTS

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In memory of S. A. Amitsur

ABSTRACT

It is shown that if the ring of constants of a restricted differential Lie algebra with a quasi-Frobenius inner part satisfies a polynomial identity (PI) then the original prime ring has a generalized polynomial identity (GPI). If additionally the ring of constants is semiprime then the original ring is PI. The case of a non-quasi-Frobenius inner part is also considered.

1. Introduction

Rings of constants of restricted differential Lie algebras with an outer action on prime and semiprime rings were investigated in detail in papers [Kh82], [Po83], [Pi86] (see also [Kh91, Ch.4, Ch6(6.4)]). In the present paper we are going to consider actions with a nontrivial inner part. In the papers [Ko91] and [Kh81] it is shown that the minimal restriction required is that the inner part should be quasi-Frobenius (selfinjective). We are interested in the structure of a prime ring R provided it is known that its ring of constants satisfies a polynomial identity. I. V. L'vov's example [Lv93] shows that in this case the ring R does not need to

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be a PI-ring. We will show that in this case R satisfies a generalized polynomial identity.

The notion of a generalized polynomial identity was introduced by S. A. Amitsur in [Am65]. In his paper S. A. Amitsur proved a structure theorem for primitive rings with generalized polynomial identities. Later W. S. Martindale [Ma69] generalized this result to arbitrary prime rings. Using this theorem we will prove that if the ring of constants is a semiprime PI-ring and the inner part is quasi-Frobenius, then the ring R is a PI-ring.

2. Preliminaries

Recall that a **derivation** of a ring R is an additive mapping d: $R \to R$ satisfying the condition $(xy)^d = x^d y + xy^d$. If d_1, d_2 are derivations then it is easy to see that the commutator $[d_1, d_2] = d_1 d_2 - d_2 d_1$ is also a derivation. Therefore the set Der R of all derivations of R is a Lie subring in the ring of endomorphisms of the abelian group $(R, +)$. Moreover, if z is a central element, then the composition of d with the multiplication by z is a derivation

$$(xy)^{dz} = z(xy)^d = (zx^d)y + x(zy^d).$$

In this case the operators of multiplication may not commute with derivations: $x^{zd} \overset{\text{def}}{=} (zx)^d = z^d x + zx^d$ or

$$zd = dz + z^d. \tag{1}$$

Thus the set Der R is a right module over the center Z. The module structure of Der R is connected with the commutator operation by the formula

$$[dz, d_1] = [d, d_1]z + dz^{d_1}. \tag{2}$$

Note that z^{d_1} is again a central element: $[z^{d_1}, x] = [z^{d_1}, x] + [z, x^{d_1}] = [z, x]^{d_1} = 0$.

Finally, if the characteristic p of the ring R is nonzero, $pR = 0$, then the pth power of any derivation will be a derivation by the Leibniz formula

$$(xy)^{d^p} = \sum_{k=0}^{k=p} C_p^k x^{d^k} y^{d^{p-k}} = x^{d^p} y + xy^{d^p}.$$

Now it is natural to formulate the following definition.

2.1. Definition: A set of derivations is called a **differential restricted Lie Z-algebra**, or in short a **Lie ∂-algebra**, if it is a right Z-submodule of $\operatorname{Der} R$ closed with respect to the operations $[d_1, d_2] = d_1 d_2 - d_2 d_1$ and $d^{[p]} = d^p$.

Note that the notion of a Lie ∂-algebra can be formalized abstractly as a restricted Lie ring with a structure of right Z-module connected with the main operations by formula (2) and the following formula:

$$(3) \qquad (dz)^{[p]} = d^{[p]} z^p + d \cdot (\cdots ((\overbrace{z^d z)^d z)^d \cdots)^d}^{p-1} z$$

which follows from (1) (see details in [Kh91, pp. 6–11]; for a slightly more general approach see in [Pa87]).

Now let R be a prime ring. Denote by $R_{\mathcal{F}}$ its left Martindale ring of quotients (see, for example, [Kh91 pp. 19–24]), by Q the symmetric Martindale ring of quotients. Recall that the center C of $R_{\mathcal{F}}$ is called the **extended** (or **generalized**) centroid of R and it is a field (see [Ma69]). All derivations of R can be uniquely extended to derivations of Q and of $R_{\mathcal{F}}$. The extended derivations are characterized in $\operatorname{Der} Q$ by the property $R^d \subseteq R$ but the linear combinations over C of extended derivations do not satisfy this property. Therefore we have to consider more general objects.

2.2. Definition: A derivation d of Q is called **R-continuous** if there exists a nonzero two-sided ideal I of R such that $I^d \subseteq R$.

It is easy to see that the set $\mathcal{D}(R)$ of all R-continuous derivations is a differential restricted Lie C-subalgebra of $\operatorname{Der} Q$.

In the present paper we consider Lie ∂-algebras of R-continuous derivations which are finite dimensional over C.

Let us fix the notations $R, C, Q, R_{\mathcal{F}}$ and $\mathcal{D}(R)$ for a prime ring, its extended centroid, the symmetric Martindale ring of quotients, the left Martindale ring of quotients and the Lie ∂-algebra of R-continuous derivations, respectively. Throughout the paper L denotes a restricted differential Lie C-algebra of R-continuous derivations, $L \subseteq \mathcal{D}(R)$, finite dimensional over C, and

$$R^L = \{r \in R \colon \forall \mu \in L \quad r^\mu = 0\}$$

is its ring of constants.

3. The inner part of a Lie ∂-algebra

If a is an element of Q then the map a^-: $x \to xa - ax$ is an R-continuous derivation, i.e. $Q^- \subseteq \mathcal{D}(R)$.

3.1. Definition: The space $K(L)$ generated over C by all $q \in Q$ such that $q^- \in L$ is called the **inner linear part of** L.

It is clear that $C^- = 0$, therefore $K(L)$ contains C and in particular it contains the unit of Q.

3.2. LEMMA: *The space $K(L)$ is a restricted Lie subalgebra of the adjoint restricted Lie algebra $Q^{(-)}$.*

Recall that $Q^{(-)}$ is a restricted Lie algebra defined on the C-space Q with the operations $[q_1, q_2] = q_1q_2 - q_2q_1$, $q^{[p]} = q^p$.

For the proof of the lemma it is enough to show that $K(L)$ is closed with respect to these operations. This fact immediately follows from the formulae

$$[a,b]^- = [a^-, b^-], \tag{4}$$

$$(a^p)^- = (a^-)^{[p]}. \tag{5}$$

3.3. LEMMA: *$K(L)^-$ is equal to the subalgebra L_{int} of all inner derivations of L.*

The proof is evident.

3.4. Definition: The associative subalgebra $\mathcal{B}(L)$ generated in Q by $K(L)$ is called the **inner associative part of** L.

3.5. LEMMA: *The algebra $\mathcal{B}(L)$ is of finite dimension over C.*

Proof: By the definition of operations in $K(L)$, the identity map id is a homomorphism of restricted Lie algebras id: $K(L) \to \mathcal{B}(L)^{(-)}$. Therefore $\mathcal{B}(L)$ as an associative envelope of $\mathcal{B}(L)^{(-)}$ is a homomorphic image of the universal restricted associative envelope $U_p(K(L))$. The latter has dimension $(\dim K(L))^p$. The lemma is proved.

3.6. LEMMA: *The algebra $\mathcal{B}(L)$ is stable under the action of L, i.e. $\mathcal{B}(L)^\mu \subseteq \mathcal{B}(L)$ for all $\mu \in L$.*

The proof follows from the formula

$$(q^\mu)^- = [q^-, \mu]. \tag{6}$$

4. Differential operators

Denote by $\Phi(L)$ the associative subring generated in the endomorphism ring of the abelian group $(Q,+)$ by L and by the operators of left and right multiplications by elements from $\mathcal{B}(L)$. By formula (1) the ring $\Phi(L)$ may not be an algebra over C. Of course $\Phi(L)$ is an algebra over the subfield of central constants

$$F = C^L \stackrel{\text{def}}{=} \{c \in C \colon \forall l \in L \ c^l = 0\}.$$

Nevertheless $\Phi(L)$ is a left and a right space over C while the subring of left multiplications, $\mathcal{B}(L)^l$, and that of right multiplications, $\mathcal{B}(L)^r$, are algebras over C.

4.1. Let us fix derivations $\mu_1, \ldots, \mu_m \in L$ such that $\mu_1 + K(L)^-, \ldots, \mu_m + K(L)^-$ form a basis for the right C-space $L/K(L)^-$. An operator Δ is called **correct** if it is of the form

$$\Delta = \mu_1^{s_1}\mu_2^{s_2}\cdots\mu_m^{s_m},$$

where $0 \le s_i < p$ and we suppose that $\mu^0 = 1$ is the identity operator.

Let U be a right linear space generated by all correct operators. By formula (1) this set will be a left space over C, also.

4.2. Proposition: *The ring $\Phi(L)$ of differential operators is isomorphic as a left and a right space over C to a tensor product over C:*

$$\Phi(L) \simeq \mathcal{L}(L)^r \otimes \mathcal{L}(L)^l \otimes U \simeq U \otimes \mathcal{B}(L)^l \otimes \mathcal{B}(L)^r, \tag{7}$$

where U is the linear space generated by correct operators over C.

Proof: It is enough to show that each differential operator $d \in \Phi(L)$ has a unique representation in the form

$$d = \sum_{i,j,k} \alpha_{ij}^{(k)} a_{ik}^r a_{jk}^l \Delta_k \tag{8}$$

and a unique representation in the form

$$d = \sum_{i,j,k} \Delta_k a_{ik}^l a_{jk}^r \alpha_{ij}^{(k)}, \tag{9}$$

where $a_{ik}, a_{jk} \in A$ and A is some fixed basis of $\mathcal{B}(L)$ over C (recall that by associativity, $a^r_{ik} a^l_{jk} = a^l_{jk} a^r_{ik}$) and the Δ_k's are correct words in $\{\mu_1, \ldots, \mu_m\}$.

The existence of this presentation follows from the relations

$$\mu a^r = a^r \mu - (a^\mu)^r, \tag{10}$$

$$\mu a^l = a^l \mu - (a^\mu)^l, \tag{11}$$

$$\mu^p = \mu_1 c_1 + \cdots + \mu_m c_m + b^r - b^l, \tag{12}$$

$$\mu_i \mu_j = \mu_j \mu_i + \mu_1 c_1 + \cdots + \mu_m c_m + b^r - b^l, \tag{13}$$

where in formula (12) $\mu_1 c_1 + \cdots + \mu_m c_m + b^-$ is a representation of $\mu^p \in L$ as a linear combination of μ_i's modulo $K(L)^-$ and in (13) $\mu_1 c_1 + \cdots + \mu_m c_m + b^-$ is the corresponding representation of $[\mu_i, \mu_j] \in L$.

The transformations of the left hand sides to the right hand sides (in the last formula only if $i > j$) allow us to reduce the operator to the form (8).

If we write formulae (10), (11) in the form

$$a^r \mu = \mu a^r + (a^\mu)^r, \tag{14}$$

$$a^l \mu = \mu a^l + (a^\mu)^l, \tag{15}$$

then in the same way the operator is reduced to the form (9).

For the proof of the uniqueness it is possible to use the following results on differential identities (see [Kh91, theorem 2.2.2, corollary 2.5.8] or [Kh78]).

4.3. PROPOSITION: *If the derivations $\mu_1, \ldots, \mu_m \in \mathcal{D}(R)$ are linearly independent modulo Q^-, and if the ring R satisfies an identity of the type*

$$\sum_{k=1}^{p^n} \sum_i a_{ki} x^{\Delta_k} b_{ki} = 0,$$

where $\Delta_1, \ldots, \Delta_{p^n}$ are all correct operators and the coefficients a_{ki}, b_{ki} belong to $R_{\mathcal{F}}$, then $\sum_i a_{ki} \otimes b_{ki} = 0$ in $R_{\mathcal{F}} \otimes_C R_{\mathcal{F}}$ for all $k, 1 \le k \le p^n$. In the same way, if the identity

$$\sum_{k=1}^{p^n} \left(\sum_i a_{ki} x b_{ki} \right)^{\Delta_k} = 0$$

is valid then $\sum_i a_{ki} \otimes b_{ki} = 0, 1 \le k \le p^n$.

Since $\mathcal{D}(I) = \mathcal{D}(R)$ and $Q(I) = Q(R)$ for each nonzero ideal I of R (see [Kh91, Lemma 1.8.4]), then Proposition 4.3 shows that the restriction of a nonzero differential operator $d \in \Phi(L)$ to I is nonzero. This note is important due to the following lemma:

4.4. LEMMA: *For each differential operator $d \in \Phi(L)$ there exists a nonzero two-sided ideal I of R such that $I^d \subseteq R$.*

The proof is easily obtained by induction from the formula $(I^2)^\mu = I^\mu I + II^\mu \subseteq I$, which is valid for the ideal I such that $I^\mu \subseteq R$.

5. Quasi-Frobenius algebras

Recall that a finite dimensional algebra B over a field C is called quasi-Frobenius if one of the following equivalent conditions is valid (see [CR62]):

(Q1): *For each left ideal λ and right ideal ρ of B the following equalities hold:*

$$l(r(\lambda)) = \lambda, \quad r(l(\rho)) = \rho,$$

where $l(A) = \{b \in B: bA = 0\}$ is the left annihilator, $r(A) = \{b \in B: Ab = 0\}$ is the right annihilator.

(Q2): *The left regular module ${}_BB$ is injective.*

(Q3): *Modules ${}_BB$ and $(B_B)^* = \mathrm{Hom}(B, C)$ have the same indecomposable components.*

Recall that for any left (right) module M the set of all linear functionals M^* has a structure of right (left) module defined by the formula $(m^*b)(m) = m^*(bm)$ (respectively $m(bm^*) = (mb)m^*$). The modules M and N for $N \simeq M^*$ are called **conjugated modules**. If the module M is of finite dimension then $(M^*)^* \simeq M$ and the conjugacy of modules (left and right), M and N, can be characterized by the existence of a nondegenerate associative bilinear form $(\ ,\ ): N \times M \to C$. In this case for every basis $a_1, \ldots, a_n$ of M there exists a *dual* basis $a_1^*, \ldots, a_n^*$ of N which is characterized by the following properties $(a_i^*, a_i) = 1, (a_i^*, a_j) = 0, i \neq j$.

Condition (Q3) implies the following condition which is important for us:

(Q4): *The sum of all right ideals ρ of B conjugated to left ideals of B is equal to B.*

It can be proved that this condition is also equivalent to B being quasi-Frobenius. Moreover, as (Q1) is left-right symmetric then the left analog of (Q4) is also valid.

(Q5): *The sum of all left ideals λ of B conjugated to right ideals of B is equal to B.*

The most important subclass of the class of quasi-Frobenius algebras is the class of Frobenius algebras. These algebras are defined by one of the following equivalent conditions ([CR62]):

(F1): *For each left ideal λ and right ideal ρ of B the following equalities hold:*

$$l(r(\lambda)) = \lambda, \quad \dim r(\lambda) + \dim \lambda = \dim B,$$
$$r(l(\rho)) = \rho, \quad \dim l(\rho) + \dim \rho = \dim B.$$

(F2): *There exists an element $\varepsilon \in B^*$ whose kernel contains no nonzero one-sided ideals of B.*

(F3): *There exists a nondegenerate associative bilinear form $B \times B \to C$.*

(F4): *The modules ${}_BB$ and $(B_B)^*$ are isomorphic.*

Classical examples of Frobenius algebras are: group algebras of finite groups over a field of arbitrary characteristic, universal restricted enveloping algebras of finite dimensional Lie p-algebras, finite dimensional Hopf algebras, Clifford algebras. Finite dimensional semisimple algebras evidently satisfy (F1), therefore they are Frobenius.

6. Universal constants

Let λ and ρ be left and right conjugated ideals of $\mathcal{B}(L)$. Let us choose a basis $a_1, \ldots, a_n$ of λ and let $a_1^*, \ldots, a_n^*$ be the dual basis of ρ. It is well-known that the element $c = \sum a_i \otimes a_i^*$ of the tensor product $\mathcal{B} \otimes_C \mathcal{B}$ commutes with the elements of $\mathcal{B}$, $bc = cb$ for all $b \in \mathcal{B}$. This implies that the set of values of the operator $c_{\lambda,\rho} = \sum a_i^l (a_i^*)^r$ is contained in the centralizer of $\mathcal{B}$. In particular, for any $\mu \in K(L)^-$ we have

$$c_{\lambda,\rho}(x)^\mu = 0. \tag{16}$$

Let $U(L)$ be the associative subring of $\Phi(L)$ generated by L and by the operators of multiplication by central elements. It is clear that $U(L)$ is both a left and a right space over C and an algebra over the field of central constants $F = C^L$.

Consider the right ideal $I = K(L)^- \cdot U(L)$ of $U(L)$. First of all the formula $\mu a^- = a^- \mu - (a^\mu)^-$ shows that I is a two-sided ideal of $U(L)$. The same formula and formulae (12), (13) show that the identity operator and operators of the form $a_1^- a_2^- \cdots a_s^- \Delta$, where Δ is a correct operator, $a_i \in K(L)$, $s \geq 0$, generate $U(L)$ as a left space over C.

6.1. PROPOSITION: *The factor-algebra $U(L)/I = \overline{U}$ is Frobenius as an algebra over $F = C^L$.*

Proof: By the well-known R. Baer theorem [Ba27] the dimension of C over F is finite and therefore $\overline{U} = U(L)/I$ has a finite dimension over F. Since $K(L)^- \subseteq I$, the elements $\bar{\mu}_1 = \mu_1 + I, \ldots, \bar{\mu}_m = \mu_m + I$ generate $\overline{U}$ as a ring over C. Moreover the relations $\bar{\mu}_i \bar{\mu}_j = \overline{[\mu_i, \mu_j]} + \bar{\mu}_j \bar{\mu}_i$ show that the images of correct words $\bar{\Delta}_k$ generate $\overline{U}$ as a left vector space over C. The main note is that the elements $\bar{\Delta}_k$ are linearly independent over C. If

$$\sum_k c_k \Delta_k = \sum_k d_k \Delta_k \in I,$$

where d_k are linear combinations of products of the type $a_1^- \cdots a_s^-$, then taking into account that $a^- = a^r - a^l$ and using Proposition 4.3, we have $c_k^r = d_k$ for all k, which is impossible since $c_k^r(1) = c_k$, $d_k(1) = 0$. Thus $\bar{\Delta}_k$ are linearly independent.

Now let us define Berkson's linear map (see [Be64]) φ: $\overline{U} \to C$ which corresponds to the element $\sum c_k \bar{\Delta}_k$, the coefficient of $\bar{\Delta}_{p^m} = \bar{\mu}_1^{p-1} \cdots \bar{\mu}_m^{p-1}$. The kernel of this linear map contains neither left nor right nonzero ideals, since the product

$$(\bar{\mu}_1^{s_1} \cdots \bar{\mu}_m^{s_m})(\bar{\mu}_1^{p-s_1-1} \cdots \bar{\mu}_m^{p-s_m-1})$$

written as a linear combination of correct words contains a unique member $\bar{\Delta}_{p^m}$ with a coefficient equal to 1.

If ψ: $C \to F$ is any projection, then the linear functional ε: $d \mapsto \psi(\varphi(d))$ satisfies (F2) and therefore $\overline{U}$ is a Frobenius algebra. The proposition is proved.

Let us consider the right subspace $\hat{U}$ of $\overline{U}$ over C generated by all nonempty words $\bar{\Delta}_k$. This space does not contain the unit (the identity operator) and it is a right (but, possibly, not a left) ideal because by formula (14) one has

$$\bar{\Delta} c \bar{\mu} = \bar{\Delta} \bar{\mu} c + \bar{\Delta} c^{\mu}.$$

By formula (13), the product $\bar{\Delta}\bar{\mu}$ can be written as a linear combination $\sum \bar{\Delta}_k c_k$, where $\bar{\Delta}_k$ are nonidentity correct operators.

Thus, the left annihilator $A = l(\hat{U})$ in the algebra $\overline{U}$ is not equal to zero. Moreover, by (F1) its dimension over F is connected with the dimension of $\hat{U}$ by the formula $\dim_F \overline{U} = \dim_F \hat{U} + \dim_F A$. On the other hand $\dim_F \overline{U} = \dim_F \hat{U} + \dim_F C$, i.e. the dimensions of A and C over F coincide. It means that A is one dimensional over C, i.e. $A = C\bar{f}$ (but possibly $A \neq \bar{f}C$ as A may not be a right C-space), where $\bar{f} = \sum \bar{\Delta}_k c_k = \sum c'_k \bar{\Delta}_k$ is a nonzero element of $\overline{U}$.

Thus, we have obtained that $\bar{f}\bar{\mu}_i = \bar{0}$ in $\overline{U}$. In the ring of differential operators this means that $f\mu_i \in K(L)^- \cdot U(L)$. We have also that $fK(L)^- \subseteq K(L)^- \cdot U(L)$ as $I = K(L)^- \cdot U(L)$ is a two-sided ideal. Thus

$$fL \subseteq f(\sum(\mu_i C + K(L)^-)) \subseteq K(L)^- \cdot U(L)$$

which, using formula (16), implies

$$((c_{\lambda,\rho}(x))^f)^\mu = 0 \tag{17}$$

for all $\mu \in L$. Let us formulate the obtained result as a lemma (see also Lemma 4.6, [Kh96]).

6.2. Lemma: *There exists a differential operator f of the type $\sum \Delta_k c_k = \sum c'_k \Delta_k$, such that for each conjugated left ideal λ and right ideal ρ of $\mathcal{B}$ with dual bases $a_1, \ldots, a_n$ and $a_1^*, \ldots, a_n^*$, the operator*

$$u_{\lambda,\rho} = \sum_i a_i^l (a_i^*)^r f \tag{18}$$

has values only in the ring of constants Q^L. There exists a nonzero ideal I of R such that

$$0 \neq u_{\lambda,\rho}(I) \subseteq R^L. \tag{19}$$

Proof: The representation of f in the form $\sum c'_k \Delta_k$ follows from (10). Formula (19) follows from formula (17), Proposition 4.3 and Lemma 4.4.

7. PI rings of constants

In this secton we will prove the theorem about a generalized polynomial identity and discuss its generalization to the case when the inner part is not quasi-Frobenius.

7.1. THEOREM: *Let L be a finite dimensional restricted differential Lie C-algebra of R-continuous derivations of a prime ring R of positive characteristic $p > 0$. Suppose that the inner associative part $\mathcal{B}(L)$ of L is quasi-Frobenius. If the ring of constants R^L is PI then R is GPI.*

Proof: Let $f(x_1, \ldots, x_n) = 0$ be a multilinear identity of R^L. Let us choose arbitrary left ideals $\lambda_1, \ldots, \lambda_n$ of $\mathcal{B}(L)$ having conjugated right ones $\rho_1, \ldots, \rho_n$. By Lemma 6.2 for every j, $1 \le j \le n$ there exists an operator

$$u_j = u_{\lambda_j, \rho_j} = \sum_i a_{ij}^l (a_{ij}^*)^r f_j = \sum_{i,k} a_{ij}^l (a_{ij}^*)^r c_k' \Delta_k$$

and a nonzero ideal I_j of R, such that $0 \ne u_j(I_j) \subseteq R^L$. If $I = \bigcap I_j$ then $u_j(x) \in R^L$ for all $x \in I$ and therefore the following differential identity holds in I:

$$f(u_1(x_1),\ u_2(x_2), \ldots,\ u_n(x_n)) = 0.$$

Let us fix some values of $x_2 = b_2, \ldots, x_n = b_n$ in I. We have

$$f\left(\sum_{i,k} (c_k' a_{i1} x_1 a_{i1}^*)^{\Delta_k}, \quad u_2(b_2), \ldots, u_n(b_n)\right) = 0. \tag{20}$$

By Leibnitz formula any expression of the type $(axb)^{\Delta}$ can be written in the form

$$(axb)^{\Delta} = ax^{\Delta}b + \sum_s a_s x^{\Delta_s} b_s,$$

where Δ_s are subwords of Δ. In particular

$$(c_k' a_{i1} x_1 a_{i1}^*)^{\Delta_k} = c_k' a_{i1} x_1^{\Delta_k} a_{i1}^* + \sum_s a_s x_1^{\Delta_s} b_s. \tag{21}$$

If Δ_{k_0} is the greatest operator such that c_{k_0}' is not zero, then this formula allows us to represent (20) in the form

$$\sum_{k=1}^{k_0} \sum_i v_{ki} x_1^{\Delta_k} w_{ki} = 0;$$

here we suppose that $\Delta_1 < \Delta_2 < \cdots < \Delta_{p^m}$ is the lexicographic ordering of all correct operators. By Proposition 4.3 applied to the prime ring I we have

$$\sum v_{k_0 i} \otimes w_{k_0 i} = 0$$

in the tensor product $I_{\mathcal{F}} \otimes_{C(I)} I_{\mathcal{F}}$, where $C(I)$ is the generalized centroid of I and $I_{\mathcal{F}}$ is the left Martindale ring of quotients of I. It is well-known and it is easy to see that $I_{\mathcal{F}} = R_{\mathcal{F}}$ and $C(I) = C(R)$. Therefore for any $x_1 \in R_{\mathcal{F}}$ we have the identity

$$\sum_i v_{k_0 i} x_1 w_{k_0 i} = 0.$$

This identity with (21) and (20) implies that the identity

$$c'_{k_0} f\left(\sum_i a_{i1} x_1 a_{i1}^*, u_2(b_2), \ldots, u_n(b_n)\right) = 0 \tag{22}$$

is valid for each $x_1 \in R_{\mathcal{F}}$.

Since the values $b_2, \ldots, b_n$ are arbitrary from I, we have an identity of the form

$$f\left(\sum_i a_{i1} x_1 a_{i1}^*, u_2(x_2), \ldots, u_n(x_n)\right) = 0, \tag{23}$$

where $x_1 \in R_{\mathcal{F}}, x_2 \in I, \ldots, x_n \in I$.

Now let us fix values $x_1 \in R_{\mathcal{F}}, x_3 = b_3 \in I, \ldots, x_n = b_n \in I$. Then in the same way we obtain

$$f\left(\sum_i a_{i1} x_1 a_{i1}^*, \sum_i a_{i2} x_2 a_{i2}^*, \ldots, u_n(x_n)\right) = 0,$$

where $x_1, x_2 \in R_{\mathcal{F}}, x_3, \ldots x_n \in I$.

Continuing this process we will obtain the following identity on $R_{\mathcal{F}}$:

$$f\left(\sum_i a_{i1} x_1 a_{i1}^*, \sum_i a_{i2} x_2 a_{i2}^*, \ldots, \sum_i a_{in} x_n a_{in}^*\right) = 0. \tag{24}$$

This is a generalized identity valid in $R_{\mathcal{F}} \supseteq R$. All we need is to prove that for some $\lambda_1, \ldots, \lambda_n; \rho_1, \ldots, \rho_n$ this is not a trivial identity. It means that the left hand side of (24) is not zero in the free product $R_{\mathcal{F}} *_C C\langle x_1, \ldots, x_n\rangle$ or, in other words, this identity does not follow from the trivial generalized identities $xc = cx$, where $c \in C$. Otherwise assume all these identities are trivial.

Any application of a trivial identity does not change the order of the indeterminates, therefore all the generalized monomials (i.e. sums of all monomials with fixed order of sequence of the indeterminates) in the identities (24) should be (trivial) identities. These generalized monomials have the form

$$\alpha_\pi \left(\sum_i a_{i\pi(1)} x_{\pi(1)} a^*_{i\pi(1)}\right) \left(\sum_i a_{i\pi(2)} x_{\pi(2)} a^*_{i\pi(2)}\right) \cdots \left(\sum_i a_{i\pi(n)} x_{\pi(n)} a^*_{i\pi(n)}\right),$$

where π is a permutation and

$$f(x_1, \ldots x_n) = \sum_\pi \alpha_\pi x_{\pi(1)} \cdots x_{\pi(n)}.$$

Since one of the coefficients α_π is equal to one (let $\alpha_1 = 1$),

$$(25) \qquad \left(\sum_i a_{i1} x_1 a^*_{i1}\right) \left(\sum_i a_{i2} x_2 a^*_{i2}\right) \cdots \left(\sum_i a_{in} x_n a^*_{in}\right) = 0.$$

Let us fix some values of $x_2, \ldots, x_n$ in R and apply Proposition 4.3 to (25), where we suppose $x = x_1$, and all coefficients a_{ki}, $k = 2, 3, \ldots p^m$ (in (25)) are zero. We have

$$\left(\sum_i a_{i1} \otimes a^*_{i1}\right) \left(\sum_i a_{i2} x_2 a^*_{i2}\right) \cdots \left(\sum_i a_{in} x_n a^*_{in}\right) = 0.$$

The set $\{a_{i1}\}$ is a basis of the ideal λ_1, i.e. this is a linearly independent set, therefore

$$a^*_{i1} \left(\sum_i a_{i2} x_2 a^*_{i2}\right) \cdots \left(\sum_i a_{in} x_n a^*_{in}\right) = 0$$

for all a^*_{i1} from the dual basis $\{a^*_{i1}\}$ of the conjugated ideal ρ_1. This implies that

$$\rho_1 \left(\sum_i a_{i2} x_2 a^*_{i2}\right) \cdots \left(\sum_i a_{in} x_n a^*_{in}\right) = 0.$$

Since the pair (λ_1, ρ_1) was chosen in an arbitrary way,

$$(26) \qquad \left(\sum_{\rho^* \simeq \text{ a left ideal of } \mathcal{B}} \rho\right) \left(\sum_i a_{i2} x_2 a^*_{i2}\right) \cdots \left(\sum_i a_{in} x_n a^*_{in}\right) = 0.$$

By Property (Q5) of quasi-Frobenius algebras

$$1 \in \mathcal{B} = \left(\sum_{\rho^* \simeq \text{ a left ideal of } \mathcal{B}} \rho\right)$$

and therefore

$$\left(\sum_i a_{i2}x_2a_{i2}^*\right)\cdots\left(\sum_i a_{in}x_na_{in}^*\right)=0.$$

Now the evident induction works. The theorem is proved.

The same proof can be applied also for some cases when the inner part is not quasi-Frobenius but has enough pairs of conjugated one-sided ideals. Indeed, let us denote by $\mathcal{B}_r$ the sum of all right ideals of a finite dimensional algebra $\mathcal{B}$ conjugated to left ones.

7.2. LEMMA: *$\mathcal{B}_r$ is a two-sided ideal of $\mathcal{B}$.*

Proof: Let ρ be a right ideal such that the dual left module $\rho^* = \mathrm{Hom}(\rho, C)$ is isomorphic to a left ideal λ. If $b \in \mathcal{B}$ then we have an exact sequence of homomorphisms of right ideals $\rho \to b\rho \to 0$. The conjugated sequence has the form $\rho^* \leftarrow (b\rho)^* \leftarrow 0$, therefore the right ideal $b\rho$ has a conjugated module $(b\rho)^*$ which is isomorphic to a left subideal of $\lambda \simeq \rho^*$. Thus $b\rho \subseteq \mathcal{B}_r$ and $\mathcal{B}_r$ is a two-sided ideal. The lemma is proved.

In the same way one can define an ideal $\mathcal{B}_l$ — the sum of all left ideals conjugated to right ones.

7.3. THEOREM: *Let L be a finite dimensional restricted differential Lie C-algebra of R-continuous derivations of a prime ring R of positive characteristic $p>0$. If the algebra of constants R^L satisfies a multilinear polynomial identity of degree n and $\mathcal{B}(L)_r^n \neq 0$, then R is a GPI-ring.*

Proof: In the same way as in the proof of Theorem 7.1 we have identities (24). If all of these identities are trivial, then we also have the identities (26) which can be written in the form

$$\mathcal{B}(L)_r\left(\sum a_{i2}x_2a_{i2}^*\right)\cdots\left(\sum a_{in}x_na_{in}^*\right)=0. \tag{27}$$

If b is an arbitrary element of $\mathcal{B}(L)$, then $b(\sum_i a_{ik}x_ka_{ik}^*) = (\sum_i a_{ik}x_ka_{ik}^*)b$. Therefore for $b \in \mathcal{B}(L)_r$, identity (27) implies

$$\left(\sum a_{i2}x_2a_{i2}^*\right)\cdots\left(\sum a_{in}x_na_{in}^*\right)b=0. \tag{28}$$

By Proposition 4.3 we have

$$\left(\sum a_{i2}\otimes a_{i2}^*\right)\cdots\left(\sum a_{in}x_na_{in}^*\right)b=0,$$

as in the proof of Theorem 7.1 we have

$$\mathcal{B}(L)_r \left(\sum a_{i3}x_3a_{i3}^*\right)\cdots\left(\sum a_{in}x_na_{in}^*\right) b = 0,$$

thus

$$\left(\sum a_{i3}x_3a_{i3}^*\right)\cdots\left(\sum a_{in}x_na_{in}^*\right) \mathcal{B}(L)_r^2 = 0.$$

Now the evident induction implies $\mathcal{B}(L)_r^n = 0$. Hence one of the GPI's (24) is not trivial. The theorem is proved.

In a symmetrical way one can prove that the condition $\mathcal{B}(L)_l^n \neq 0$ also implies that one of the identities (24) is not trivial. It can be proved that $\mathcal{B}_r^n = 0$ iff $\mathcal{B}_l^n = 0$:

7.4. PROPOSITION: *Let $\mathcal{B}$ be a finite dimensional algebra. Then all $n+1$ conditions $\mathcal{B}_r^k\mathcal{B}_l^{n-k} = 0$, $k = 0, \ldots, n$ are equivalent to each other.*

Proof: It is enough to show that the conditions for k and $k+1$ are equivalent. The condition $\mathcal{B}_r^k\mathcal{B}_l^{n-k} = 0$ is equivalent to $\mathcal{B}_r^k\mathcal{B}_l^{n-k-1}\lambda = 0$ for all pairs of conjugated ideals ρ, λ. Since the form $(\ ,\ )$: $\rho \times \lambda \to C$ is nondegenerate, the last condition for given λ, ρ is equivalent to $(\rho, \mathcal{B}_r^k\mathcal{B}_l^{n-k-1}\lambda) = 0$. By associativity of the form this is equivalent to $(\rho\mathcal{B}_r^k\mathcal{B}_l^{n-k-1}, \lambda) = 0$, and since the form is nondegenerate this is equivalent to $\rho\mathcal{B}_r^k\mathcal{B}_l^{n-k-1} = 0$. The last conditions for all pairs of conjugated ideals λ, ρ are equivalent to $\mathcal{B}_r^{k+1}\mathcal{B}_l^{n-k-1} = 0$. The proposition is proved.

Now it is a question of interest whether the condition $\mathcal{B}(L)_r^n = 0$ implies that all identities (24) are trivial generalized polynomial identities. The answer is yes:

7.5. PROPOSITION: *If under the conditions of Theorem 7.3, $\mathcal{B}(L)_r^n = 0$, then all identities (24) are trivial.*

Proof: It is enough to show that all the generalized monomials (25) are trivial identities. We will prove by inverse induction on k that for arbitrary $b_1, \ldots, b_k \in \mathcal{B}(L)_r$ the generalized polynomial

$$\left(\sum_i a_{i\ k+1}x_{k+1}a_{i\ k+1}^*\right)\cdots\left(\sum_i a_{in}x_na_{in}^*\right) b_kb_{k-1}\cdots b_1 = 0 \tag{29}$$

is a trivial generalized identity.

If $k = n$, then (29) has the form $b_nb_{n-1}\cdots b_1 = 0$ that is a trivial identity as $\mathcal{B}(L)_r^n = 0$.

Assume that (29) is a trivial identity. The identities

$$b\left(\sum_i a_{is}xa_{is}^*\right) = \left(\sum_i a_{is}xa_{is}^*\right)b, \quad b \in \mathcal{B}(L) \tag{30}$$

are trivial generalized polynomial identities (as well as any linear generalized identity). Let $b_k = a_{ik}^*$, then from (29) and (30) we have the following trivial identity:

$$a_{ik}^*\left(\sum_i a_{i\ k+1}x_{k+1}a_{i\ k+1}^*\right)\cdots\left(\sum_i a_{in}x_na_{in}^*\right)b_{k-1}\cdots b_1 = 0.$$

Multiplication of this equality on the left by $a_{ik}x_k$ and summation over i gives the equality (29) with a smaller k. The proposition is proved.

8. Semiprime PI-rings of constants

In this secton we will prove, under the conditions of Theorem 7.1, that if the ring of constants R^L is a semiprime PI-ring, then R is also PI.

8.1. THEOREM: *Let L be a finite dimensional restricted differential Lie C-algebra of R-continuous derivations of a prime ring R of positive characteristic $p > 0$. Suppose that the inner associative part $\mathcal{B}(L)$ of L is quasi-Frobenius. If the ring of constants R^L is a semiprime PI-ring, then R is PI.*

Proof: By Theorem 7.1 the ring R satisfies a generalized polynomial identity. Moreover all generalized polynomial identities (24) hold in its left Martindale ring of quotients $R_{\mathcal{F}}$. In particular they hold in the central closure $RC \subseteq R_{\mathcal{F}}$ of the ring R. By the Martindale structure theorem [Ma69] this central closure has an idempotent e, such that $D = eRCe$ is a skew field of finite dimension over C. (Note that formally the Martindale theorem can be applied only if the coefficients of the identity belong to R. In our case they belong to $R_{\mathcal{F}}$ but may not belong to R. Nevetheless Martindale's original proof is correct for our case too; see, for instance, [Kh91, Theorem 1.13.4] or the special investigation in [La86].)

Thus, by the Martindale theorem, RC is a primitive ring with a nonzero socle. The N. Jacobson structure theorem [Ja64] shows that RC is a dense subring in the finite topology in the complete ring $\mathcal{E}$ of linear transformations of the left space $V = eRC$ over the skew field D.

Moreover, the left Martindale quotient ring $(RC)_{\mathcal{F}}$ is equal to $\mathcal{E}$ (see [Ha82, Lemma 1.1] and [Ha87, Remark 4.9] or [Kh91, Theorem 1.15.1]). It is easy to see that $R_{\mathcal{F}} \subseteq (RC)_{\mathcal{F}} = \mathcal{E}$. (Indeed, if $q \in R_{\mathcal{F}}$ and $Iq \subseteq R$ for a nonzero ideal I of R, then we can extend q to the ideal IC of RC by the obvious formula $(\sum i_\alpha c_\alpha)q = \sum (i_\alpha q)c_\alpha$. This is well-defined. Indeed, if $\sum i_\alpha c_\alpha = 0$ and J is a nonzero ideal of R such that $Jc_\alpha \subseteq R$, then $\sum (jc_\alpha)i_\alpha = 0$ for all $j \in J$. Therefore $\sum (jc_\alpha)(i_\alpha q) = 0$; i.e. $J(\sum c_\alpha(i_\alpha q)) = 0$ and $\sum (i_\alpha q)c_\alpha = 0$.) Now all the coefficients of (24) belong to $\mathcal{E}$, and since addition and multiplication are continuous in the finite topology, the identities (24) hold in $\mathcal{E}$. (Here one can use also Corollary 2.3.2 from [Kh91] which allows us to extend identities from RC to $(RC)_{\mathcal{F}}$.)

Now we are going to prove that the space V is finite dimensional over D. In that case the dimension of $\mathcal{E}$ over C will also be finite: $d = \dim_C \mathcal{E} = (\dim_D V)^2 \cdot \dim_C D$ and $\mathcal{E}$ (and therefore R), like any d-dimensional algebra, will satisfy the standard polynomial identity:

$$S_d(x_1, \ldots, x_{d+1}) \equiv \sum (-1)^\pi x_{\pi(1)} \cdots x_{\pi(d+1)} = 0.$$

On the contrary, suppose that V has infinite dimension $\dim V = \beta$. Let M be the set of all linear transformations whose rank is less then β. (Recall that the **rank** of a transformation l is the dimension over D of its image.) It is well-known that M is a maximal ideal of $\mathcal{E}$. So the factor ring $\bar{\mathcal{E}} = \mathcal{E}/M$ is a simple ring with a unit.

8.2. LEMMA: *The ring $\bar{\mathcal{E}}$ is not Artinian.*

Proof: Let $\{e_i, i \in I\}$ be a basis of V over D, and

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

be a chain of subsets such that $|I_k \setminus I_{k+1}| = \beta$, and let

$$A_n = \{l \in \mathcal{E}\colon e_i l = 0 \ \ \forall i \in I \setminus I_n\}.$$

Then

$$(A_1 + M)/M \supset A_2 + M/M \supset \cdots \supset A_n + M/M \supset \cdots$$

is an infinite descending chain of right ideals of $\bar{\mathcal{E}}$.

Indeed, if $A_n + M = A_{n+1} + M$, then for the transformation w, defined by

$$e_i w = \begin{cases} e_i & \text{if } i \in I_n \setminus I_{n+1}, \\ 0 & \text{otherwise,} \end{cases}$$

we should get a presentation $w = a + m$, where $a \in A_{n+1}$, $m \in M$. Let V_1 be a subspace generated by $\{e_i: i \in I_n \setminus I_{n+1}\}$. Then $V_1 = V_1 w \subseteq V_1 a + V_1 m = V_1 m$. However, $\dim_D V_1 = \beta$, while $\dim_D V_1 m \leq \dim V m < \beta$, which is a contradiction. The lemma is proved.

8.3. LEMMA: *The ring $\bar{\mathcal{E}}$ does not satisfy a nontrivial generalized polynomial identity.*

Proof: Like any simple ring with a unit, the ring $\bar{\mathcal{E}}$ is primitive. If it satisfies a GPI, then by the S.A. Amitsur structure theorem [Am65] it has a nonzero socle S, which is a two-sided ideal and therefore $S = \bar{\mathcal{E}}$. In the N. Jacobson presentation of $\bar{\mathcal{E}}$ as a dense ring of linear transformations, the socle consists of all transformations of finite rank. This means that the unit has finite rank and therefore the space has finite dimension. Thus $\bar{\mathcal{E}}$ is the ring of all linear transformations of a finite dimensional space over a skew field. In particular $\bar{\mathcal{E}}$ is Artinian; this is a contradiction to Lemma 8.2. The lemma is proved.

Let us consider now identities (24). We have seen that all these identities hold in $\mathcal{E}$. If we apply the natural homomorphism $\varphi: \mathcal{E} \to \bar{\mathcal{E}} = \mathcal{E}/M$ we obtain the following identities of the ring $\bar{\mathcal{E}}$:

$$f\left(\sum_i \bar{a}_{i1} x_1 \bar{a}_{i1}^*, \ldots, \sum_i \bar{a}_{in} x_n \bar{a}_{in}^*\right) = 0, \tag{31}$$

where $\bar{a} = \varphi(a) = a + M$.

By Lemma 8.3 all we need is to prove that one of the identities (31) is a nontrivial GPI of $\bar{\mathcal{E}}$.

First of all we have to calculate the generalized centroid of $\bar{\mathcal{E}}$. As $\bar{\mathcal{E}}$ is a simple ring with a unit, it equals its left Martindale quotient ring and therefore the generalized centroid is equal to the center.

8.4. LEMMA: *The center of $\bar{\mathcal{E}}$ is canonically isomorphic to C, $C(\bar{\mathcal{E}}) = \varphi(C)$.*

Proof: See [Ro58, Corollary 3.3].

We will need the following result which gives a criterion for determining when the ring of constants is semiprime (see Theorem 5.1 [Kh96]).

8.5. THEOREM: *Under the conditions of Theorem 8.1, the ring of constants is semiprime if and only if $\mathcal{B}(L)$ is differentially semisimple, i.e. it has no nonzero differential (with respect to action of L) ideals with zero multiplication or, equivalently, it is a sum of a finite number of differentially simple algebras.*

By this theorem we have that in our situation the algebra $\mathcal{B}(L)$ is differentially semisimple.

8.6. LEMMA: *The ideal M is a differential ideal with respect to L, i.e. $M^\mu \subseteq M$ for each $\mu \in L$.*

Proof: Note that M is a differential ideal with respect to each derivarion of $\mathcal{E}$. Indeed, if $l \in M$ then l is a transformation of rank less than β and the projection $e: V \to \text{im} l$ also has rank less than β, in which case $l = le$. We have $l^\mu = l^\mu e + le^\mu \in M$ for each derivation $\mu \in \text{Der}(\mathcal{E})$.

By proposition 1.8.1 [Kh91] any R-continuous derivation has a unique extension to $R_\mathcal{F}$. In particular each derivation from L is defined on RC. Again by the same proposition we have that the elements of L have unique extensions to $(RC)_\mathcal{F} = \mathcal{E}$. Thus we have obtained that the ideal M is differential with respect to L. The lemma is proved.

As a consequence we have that the intersection $M_0 = M \cap \mathcal{B}(L)$ is a differential ideal of $\mathcal{B}(L)$, which is not equal to $\mathcal{B}(L)$ (it does not contain 1). The left annihilator $l(M_0)$ of M_0 in $\mathcal{B}(L)$ is also a differential ideal, therefore $l(M_0) \cap M_0$ is a differential ideal with zero mulitiplication. By Theorem 8.5, $l(M_0) \cap M_0 = 0$. In the same way the left annihilator of the sum $l(M_0) + M_0$ is zero (it is contained in $l(M_0)$ and, therefore, has a zero multiplication). Now property (Q1) of quasi-Frobenius algebras implies that $l(M_0) + M_0 = r(l(l(M_0) + M_0)) = r(0) = \mathcal{B}(L)$ and, finally

$$\mathcal{B}(L) = l(M_0) \oplus M_0 = e\mathcal{B}(L) \oplus (1 - e)\mathcal{B}(L), \tag{32}$$

where e is a central idempotent defined by the corresponding decomposition of the unit $1 = e \oplus (1 - e)$.

Let us return to identities (31). Suppose that in these identities $\{a_{ij}\}$ and $\{a^*_{ij}\}$ are bases of conjugated ideals λ_j, ρ_j contained in $l(M_0)$. In that case the sets $A_j = \{\bar{a}_{ij}, i = 1, \dots m\}$ are linearly independent over the center of $\bar{\mathcal{E}}$ (see Lemma 8.4). Moreover, the C-space generated by all possible a^*_{ij}'s contains the unit e of $l(M_0)$ because for each conjugated pair of ideals λ, ρ the one-sided ideals

$e\lambda, e\rho$ are also conjugated with respect to the same form (note that e is a central idempotent of $\mathcal{B}(L)$). This implies that the linear space over the center of $\bar{\mathcal{E}}$ generated by all $\bar{a}_{ij}^*$'s contains the unit $\bar{e}$ of $\bar{\mathcal{E}}$. This fact allows us to prove that one of the identities (31) is nontrivial in the same manner as was done at the end of the proof of Theorem 7.1. By Lemma 8.3, Theorem 8.1 is proved.

In this proof we used the fact that the inner part $\mathcal{B}(L)$ is differentially semisimple and that it has enough pairs of conjugated ideals. Therefore in the way analogous to Theorem 7.3 we can formulate a slightly more general result.

8.7. THEOREM: *Let L be a finite dimensional restricted differential Lie C-algebra of R-continuous derivations of a prime ring R of positive characteristic $p > 0$. Suppose that the inner part $\mathcal{B}(L)$ is a direct sum of differentially simple ideals*

$$\mathcal{B}(L) = B_1 \oplus B_2 \oplus \cdots \oplus B_m.$$

If the algebra of constants R^L satisfies a multilinear polynomial identity of degree n and $(B_i)_r^n \neq 0, i = 1, \ldots, m$, then R is a PI-ring.

The only place where we have used that $\mathcal{B}(L)$ is quasi-Frobenius is decomposition (32). Therefore it is enough to show that each differential ideal of the direct sum of differentially simple algebras with units is a direct summand. If $B = B_1 \oplus B_2 \oplus \cdots \oplus B_m$ is a direct sum of differentially simple algebras, then for any differential ideal A we have that AB_i is a differential ideal of B_i. This implies that either $B_i \subseteq A$ or $AB_i = 0$. In the same way either $B_i \subseteq A$ or $B_iA = 0$. Let $l(A)$ be the left annihilator of A; then $l(A) \cap A$ is a differential ideal with zero multiplication, so its product with each B_i is zero. This is possible only if the intersection is zero. In the same way the left annihilator of the sum $l(A) + A$ has a zero multiplication and therefore it is equal to zero. It means that $l(A) + A$ contains all the components B_i and $l(A) \oplus A = B$.

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